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Non-linear Approximation and Interpolation Spaces

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We study n-term wavelet-type approximations in Besov and Triebel–Lizorkin spaces. In particular, we characterize spaces of functions which have prescribed degree of n-term approximation in terms of interpolation spaces. These results are applied to identify interpolation spaces between Triebel–Lizorkin and Besov spaces.

Key Words: non-linear approximation; interpolation of operators; wavelets.

1. INTRODUCTION

In recent years there has been great interest in n-term wavelet approximation. This is basically due to the numerical applications of wavelets to such fields as statistics, signal and image processing, and the numerical solution of PDE's.

In order to describe the results of this paper we first recall, without elaboration, the usual setting of wavelet decompositions. For a detailed discussion we refer the reader to Meyer [M] and Daubechies [Da]. We would like to point out that our results can be established for quite general wavelet-type decompositions, however for notational reasons we shall restrict ourselves to orthogonal wavelet bases and indicate the necessary changes in order to deal with more general bases.

Throughout this paper, we denote by $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^d$ and by $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions. We also denote by $\mathscr{S}'/\mathscr{P}$ the space of equivalence classes of distributions in $\mathscr{S}'$ modulo polynomials, i.e., $\mathscr{S}'/\mathscr{P}$ is the dual of the space $\mathscr{S}_\infty := \mathscr{S}_\infty(\mathbb{R}^d)$ of all functions $\eta \in \mathscr{S}$ such that $\int \eta(x)\, x^\alpha\, dx = 0$ for $\alpha \in \mathbb{Z}_+^d$, ($\mathbb{Z}_+ := \{n: n \geqslant 0\}$).

We write $\mathscr{D}$ for the family of all dyadic cubes in $\mathbb{R}^d$ and $\mathscr{D}_m$, $m \in \mathbb{Z}$, for the collection of all cubes $I \in \mathscr{D}$ of volume $|I| = 2^{-dm}$. Finally, for any

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distribution $f \in \mathscr{S}'$ and any dyadic cube $I = 2^{-j}[k, k+1]$, $k \in \mathbb{Z}^d$, $j \in \mathbb{Z}^d$ we define the distribution

$$f_I(x) := 2^{jd/2} f(2^j x - k), \tag{1.1}$$

where the dilation and translation are considered in distributional sense.

Multivariate wavelet bases are typically constructed as tensor products of a univariate scaling function $\psi^0 := \phi$ and n associated wavelet ψ. Let V denote the set of nonzero vertices of the unit cube in $\mathbb{R}^d$, for each vertex $v = (v_1, \ldots, v_d) \in V$ we set

$$\psi^v(x) := \psi^{v_1}(x_1) \cdots \psi^{v_d}(x_d),$$

and we define $\Psi := \{\psi^v : v \in V\}$. Then, the collection

$$W := \{\psi_I^v : I \in D, v \in V\}$$

forms an orthonormal basis for the space $L_2(\mathbb{R}^d)$. We refer the reader to [Da] for details on the construction of Ψ.

Typically one requires that for some $r \geqslant 1$ the wavelet set Ψ satisfies $\Psi \subset C^r(\mathbb{R}^d)$ and

$$\begin{aligned} &\text{(i)} \quad |(\psi^v)^{(\alpha)}(x)| \leqslant C(1+|x|)^{-M}, && |\alpha| \leqslant r, \quad v \in V, \\ &\text{(ii)} \quad \int_{\mathbb{R}^d} x^\alpha \psi^v(x)\, dx = 0, && |\alpha| < r, \quad v \in V. \end{aligned} \tag{1.2}$$

If $f \in \mathscr{S}'/\mathscr{P}$ then for r, M sufficiently large (depending on the order of f) we define the wavelet coefficients

$$a_I^v(f) := \langle f, \psi_I^v \rangle, \qquad I \in D, v \subset V,$$

and we set

$$a_I(f) := \left(\sum_{v \in V} (a_I^v(f))^2 \right)^{1/2}.$$

For every $f \in \mathscr{S}'/\mathscr{P}$ and $0 < p \leqslant \infty$ we also define the functions

$$S_p^\alpha(f, x) := \left(\sum_{I \in D} (|I|^{-\alpha/d - 1/2} a_I(f)\, \chi_I(x))^p \right)^{1/p}, \qquad x \in \mathbb{R}^d,$$

where χ_I is the characteristic function of I, and

$$T_p^\alpha(f, m) := \left(\sum_{I \in D_m} (|I|^{-\alpha/d+1/p-1/2} a_I(f))^p \right)^{1/p}, \qquad m \in \mathbb{Z},$$

with the usual modifications for $p = \infty$.

By varying the smoothness and decay parameters r and M, one can prove that W forms an unconditional basis for a host of distribution spaces such as the homogeneous Triebel–Lizorkin and Besov spaces, $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ respectively. We say that a wavelet set Ψ is admissible for the triplet (s, p, q), $s \in \mathbb{R}$, $0 < p,\ q \leqslant \infty$ if $r > \max\{J-d-s, s\}$ and $M > \max\{J, d+r\}$ where $J := d/\min\{1, p, q\}$. These assumptions guarantee that W is an unconditional basis for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ (see [M], [K1], and [FJW]). We recall also the so-called Meyer wavelet set [M] which satisfies (1.2) for any choice of the parameters r and M.

Besov spaces were initially introduced by Besov in the late 1950s by means of the modulus of smoothness and were closely related to Approximation Theory. At the end of 1960s, new methods using the Fourier Transform were employed that led to alternative descriptions of these spaces and to the introduction of the Triebel–Lizorkin spaces. In this article however, we prefer to define these spaces directly and in a unified way by means of wavelet decompositions. These definitions are equivalent to the original ones. In particular, if Ψ is admissible for (s, p, q), $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leqslant \infty$, $\dot{F}^s_{p,q}$ is defined as the set of all distributions in $\mathcal{S}'/\mathcal{P}$ for which the following (quasi)-norm

$$\|f\|_{\dot{F}^s_{p,q}} := \|S^s_q(f, \cdot)\|_{L_p(\mathbb{R}^d)} \tag{1.3}$$

is finite. Moreover, every $f \in \dot{F}^s_{p,q}$ has the unique wavelet decomposition

$$f = \sum_{I \in D} A_I(f), \qquad A_I(f) := \sum_{v \in V} a_I^v \psi_I^v, \tag{1.4}$$

where the convergence is considered in the sense of $\mathcal{S}'/\mathcal{P}$ (see [T]).

Similarly, if Ψ is admissible for (s, p, q), where $s \in \mathbb{R}$ and $0 < p,\ q \leqslant \infty$, then $\dot{B}^s_{p,q}$, is defined as the set of all distributions in $\mathcal{S}'/\mathcal{P}$ for which the following (quasi)-norm

$$\|f\|_{\dot{B}^s_{p,q}} := \|T^s_p(f, \cdot)\|_{\ell_q(\mathbb{Z})} \tag{1.5}$$

is finite. In addition, every $f \in \dot{B}^s_{p,q}$, enjoys the representation (1.4).

We note that the above definitions are independent of admissible wavelet sets since different sets give rise to equivalent (quasi)-norms. Also it is easily seen from the definitions that $\dot{F}^s_{p,p} = \dot{B}^s_{p,p}$. To avoid any confusion, for the rest of the paper, if not otherwise mentioned, we consider Meyer's wavelet set which is admissible for all values of s, p and q. However, as we already mentioned, our results hold for more general bases: the main requirements for our subsequent analysis are (1.3) and (1.5) which are known to hold for instance, for biorthogonal wavelets, prewavelets or the more general decompositions considered in [KP].

We recall that, for $1 < p < \infty$, $\dot{F}^0_{p,2} = L_p$ while, for $0 < p \leqslant 1$, $\dot{F}^0_{p,2} = H_p$ the real Hardy spaces. Also, for $s > 0$, $1 < p < \infty$, $\dot{F}^s_{p,2} = H^s_p$ the potential space, and for integer values of s, $\dot{F}^s_{p,2}$ is the usual Sobolev space W^s_p equipped with its seminorm (see [T]). The characterization (1.5) of Besov spaces has been proved by several authors in various subcases (see [M], [K1], and the references therein) under different assumptions. For the Triebel–Lizorkin spaces, we refer the reader to [M] and [FJW].

Let Σ_n be the set of all functions

$$S = \sum_{I \in \Lambda} A_I(S),$$

where $\Lambda \subset \mathcal{D}$ is a set a dyadic cubes with cardinality $\#\Lambda \leqslant n$.

We are interested in approximating in the Triebel–Lizorkin or Besov spaces by the elements of Σ_n, $n \in \mathbb{Z}_+$. For a given distribution f and any quasi-normed subspace $X \subset \mathcal{S}'/\mathcal{P}$ we define

$$\sigma_n(f, X) := \inf_{S \in \Sigma_n} \|f - S\|_X.$$

We shall consider the cases where $X = \dot{F}^s_{p,q}$ or $X = \dot{B}^s_{p,q}$, $s \in \mathbb{R}$, $p, q > 0$.

Our main goal is to characterize the functions f for which $\sigma_n(f, X)$ has a prescribed rate of decay as $n \to \infty$. For $0 < q \leqslant \infty$ and $\alpha > 0$ we define the approximation class $\mathcal{A}^\alpha_q(X)$ to be the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$|f|_{\mathcal{A}^\alpha_q(X)} = \begin{cases} \left(\sum_{j=0}^{\infty} [2^{j\alpha} \sigma_{2^j}(f)_X]^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{j \geqslant 0} 2^{j\alpha} \sigma_{2^j}(f)_X, & q = \infty, \end{cases} \tag{1.6}$$

is finite.

Let now S_0 be any near-best approximant to f from $\Sigma_1 \subset X$ (i.e., $\|f - S_0\|_X \leqslant 2\sigma_1(f)_X$), we further define

$$\|f\|_{\mathcal{A}^\alpha_q(X)} := \|S_0\|_X + |f|_{\mathcal{A}^\alpha_q(X)}. \tag{1.7}$$

We note that $\|f\|_{\mathcal{A}^\alpha_q(X)}$ is independent of the choice of S_0. Indeed, if S_0 and S'_0 are two different near-best approximants to f from Σ_1 then

$$\|S_0\|_X \leqslant C\{\|S_0 - S'_0\|_X + \|S'_0\|_X\} \leqslant C\{\sigma_1(f)_X + \|S'_0\|_X\}. \tag{1.8}$$

Moreover, if $\alpha_1 > \alpha_2$ and $0 < q_1,\ q_2 \leqslant \infty$, we have the continuous embedding $\mathscr{A}^{\alpha_1}_{q_1}(X) \subset \mathscr{A}^{\alpha_2}_{q_2}(X)$.

We are going to characterize the spaces $\mathscr{A}^{\alpha}_{q}(X)$ in terms of interpolation spaces and in terms of smoothness spaces. In the literature complete characterizations are known for the cases $X = H_p = \dot{F}^0_{p,2}$, $0 < p < \infty$, $X = \dot{B}^0_{p,p}$, $0 < p < \infty$ and $X = H^s_2 = \dot{F}^s_{2,2}$, $s \in \mathbb{R}$. The spaces $\mathscr{A}^{\alpha/d}_{q}(H_p)$, $0 < \alpha$, $p < \infty$, $0 < q \leqslant \infty$, have been investigated by DeVore, Jawerth and Popov [DJP] where they established that if $0 < \alpha < s$ and $1/\tau = s/d + 1/p$ then

$$\mathscr{A}^{\alpha/d}_{q}(H_p) = (H_p,\ \dot{B}^s_{\tau,\tau})_{\alpha/s,\,q},$$

where $(H_p,\ \dot{B}^s_{\tau,\tau})_{\alpha/s,\,q}$ is the real interpolation space between H_p and $\dot{B}^s_{\tau,\tau}$ (see Section 2). In the special case where $1/q = \alpha/d + 1/p$ they identified $\mathscr{A}^{\alpha/d}_{q}(H_p)$ with the Besov space $\dot{B}^{\alpha}_{q,q}$, in view of the fact that

$$(H_p,\ \dot{B}^s_{\tau,\tau})_{\alpha/s,\,q} = \dot{B}^{\alpha}_{q,q}.$$

Approximation in $\dot{B}^{\alpha}_{p,p}$ was considered by Cohen, DeVore and Hochmuth, due to the simple structure of the space and the advantages that this presents in various applications. If $\alpha > 0$ and $0 < q \leqslant \infty$, they proved in [CDH] that:

$$\mathscr{A}^{\alpha/d}_{q}(\dot{B}^0_{p,p}) = \mathscr{A}^{\alpha/d}_{q}(H_p).$$

Approximation in the Hilbertian Sobolev space $H^s_2(\Omega)$, $s \in \mathbb{R}$, has been considered by Dahlke, Dahmen and DeVore [DDD] in a slightly different setting. They actually proved that if Ω is a bounded, open and connected Lipschitz domain in $\mathbb{R}^d$, $\alpha > 0$ and $1/q = \alpha/d + 1/2$ then

$$\mathscr{A}^{\alpha/d}_{q}(H^s_2(\Omega)) = B^{s+\alpha}_{qq}(\Omega),$$

where $H^s(\Omega)$ and $B^{s+\alpha}_{qq}(\Omega)$ are the corresponding non-homogeneous spaces defined on Ω. In the case $\Omega = \mathbb{R}^d$ we note that this result can be easily modified to show that

$$\mathscr{A}^{\alpha/d}_{q}(\dot{F}^s_{2,2}) = \dot{B}^{s+\alpha}_{q,q}.$$

Our goal is to extend these results to the full range of Triebel–Lizorkin and Besov spaces defined on $\mathbb{R}^d$. In particular we shall prove the following results:

THEOREM 1.9. *Let* $0<p<\infty$, $0<q$, $t\leqslant\infty$, $\alpha>0$ *and* $\beta\in\mathbb{R}$. *If* $\alpha<\gamma-\beta$ *and* τ *is defined by* $1/\tau=(\gamma-\beta)/d+1/p$ *then*

$$A_q^{\alpha/d}(\dot{F}^{\beta}_{p,t})=(\dot{F}^{\beta}_{p,t},\dot{B}^{\gamma}_{\tau,\tau})_{\frac{\alpha}{\gamma-\beta},q}. \tag{1.10}$$

We note that if instead of using Meyer wavelets one defines the approximation spaces by means of a more general wavelet basis Ψ then, the above theorem still holds as long as Ψ is admissible for (β, p, t) and (γ, τ, τ).

We shall also prove that as far as n-term wavelet approximation is concerned the third index t, in the spaces $\dot{F}^{\beta}_{p,t}$ is irrelevant.

THEOREM 1.11. *Let* $0<p<\infty$, $0<q\leqslant\infty$, $\alpha>0$ *and* $\beta\in\mathbb{R}$. *Then, for any* $t_1, t_2>0$

$$A_q^{\alpha/d}(\dot{F}^{\beta}_{p,t_1})=A_q^{\alpha/d}(\dot{F}^{\beta}_{p,t_2}). \tag{1.12}$$

Moreover, in the special case where $1/q=\alpha/d+1/p$ *then*

$$\mathscr{A}_q^{\alpha/d}(\dot{F}^{\beta}_{p,t})=\dot{B}^{\beta+\alpha}_{q,q}. \tag{1.13}$$

Also as a consequence of Theorems 1.9 and 1.11 we get the following interpolation result:

COROLLARY 1.14. *Let* $0<p<\infty$, $0<t\leqslant\infty$, *and* $\beta<\gamma$. *If* $1/\tau=(\gamma-\beta)/d+1/p$ *and* $0<\theta<1$, *then*

$$(\dot{F}^{\beta}_{p,t},\dot{B}^{\gamma}_{\tau,\tau})_{\theta,q}=\dot{B}^{\beta+\alpha}_{q,q}, \tag{1.15}$$

where $\alpha=\theta(\gamma-\beta)$ *and* $1/q=\alpha/d+1/p$.

Formula (1.15) appears to be new; in the literature (see [T], [T1]) one usually finds interpolation results involving pairs of either Besov spaces or Triebel–Lizorkin spaces which are more tractable.

Considering approximation within the scale of Besov spaces we shall also establish a similar characterization for the $A_q^{\alpha/d}(\dot{B}^{\beta}_{p,t})$ spaces:

THEOREM 1.16. *Let* $0<p<\infty$, $0<q$, $t\leqslant\infty$, $\alpha>0$ *and* $\beta\in\mathbb{R}$. *If* $\alpha<\gamma-\beta$ *and* τ, r *are defined by* $1/\tau-1/p=1/r-1/t=(\gamma-\beta)/d$ *then*

$$A_q^{\alpha/d}(\dot{B}^{\beta}_{p,t})=(\dot{B}^{\beta}_{p,t},\dot{B}^{\gamma}_{\tau,r})_{\frac{\alpha}{\gamma-\beta},q}.$$

We note that contrary to the case of Theorem 1.9 the third index t in the Besov spaces $\dot{B}^{s}_{p,t}$ is not a free variable anymore and r, t are related in the same way that τ depends on p.

An outline of the paper is as follows: In Section 2 we give a short description of the basic theory of interpolation spaces. In Sections 3 and 4 we prove Jackson and Bernstein-type inequalities. In Section 5 we state the proofs of Theorems 1.9 and 1.16 and in Section 6 we give the proofs of Theorem 1.11 and Corollary 1.14.

2. THE K-METHOD AND INTERPOLATION SPACES

The real method of interpolation provides a means of extending the classical theorem of Marcinkiewicz on interpolation between L_p-spaces to more general Banach spaces. Based on the K-functional introduced by Petree and Lions this method is intrinsically connected to approximation theory and appears naturally in the study and characterization of approximation spaces.

Let X, Y be a pair of subspaces of $\mathcal{S}'/\mathcal{P}$. For every $f \in X+Y$ and $t>0$ the K-functional is defined by

$$K(f,t) := K(f,t,X,Y) := \inf_{f=g+h} \|g\|_X + t\,\|h\|_Y.$$

For every $0<\theta<1$, and $0<q\leqslant\infty$, the interpolation space $(X,Y)_{\theta,q}$ is defined as the set of all $f \in X+Y$ such that

$$|f|_{(X,Y)_{\theta,q}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} [2^{j\theta}K(f,2^{-j})]^q\right)^{1/q}, & 0<q<\infty, \\ \sup_{j\geqslant 0} 2^{j\theta}K(f,2^{-j}), & q=\infty \end{cases}$$

is finite.

We will be interested in the cases where the role of X and Y will be played by the Triebel–Lizorkin and Besov spaces. The connection between interpolation and the approximation spaces defined in (1.6) is best illustrated by the following result:

THEOREM 2.1. *Let $0<\gamma<r$ and $0<q\leqslant\infty$. We also assume that for every $n\in\mathbb{Z}_+$ and $S\in\Sigma_n$ the following two fundamental inequalities hold:*

(2.2) **Jackson:** $$\sigma_n(f,X) \leqslant Cn^{-r/d}\,\|f\|_Y,$$

and

(2.3) **Bernstein:** $$\|S\|_Y \leqslant Cn^{r/d}\,\|S\|_X.$$

Then,

(2.4) $$(X,Y)_{\gamma/r,q} = A_q^{\gamma/d}(X).$$

Proof. The proof of the theorem is well known and we refer the reader to [DL] for details, we mention only that the proof actually shows that there exist C_1, such that for every $f \in \mathscr{A}_q^{\gamma/d}(X)$

$$|f|_{(X,Y)_{\gamma/r,q}} \leqslant C_1 \, \|f\|_{\mathscr{A}_q^{\gamma/d}(X)},$$

and C_2 such that for every $f \in (X, Y)_{\gamma/r,q}$

$$|f|_{\mathscr{A}_q^{\gamma/d}(X)} \leqslant C_2 \, |f|_{(X,Y)_{\gamma/r,q}}. \qquad \blacksquare$$

3. JACKSON-TYPE INEQUALITIES

In this section we are going to establish Jackson-type inequalities for the various spaces we are interested in. We start with approximation in the space $\dot{F}^{\beta}_{p,t}$, $\beta \in \mathbb{R}$, $p, t > 0$.

THEOREM 3.1. *Let $0 < p < \infty$, $0 < t \leqslant \infty$ and $\beta < \gamma$. If τ is defined by $1/\tau = (\gamma - \beta)/d + 1/p$. Then, for every $f \in \dot{B}^{\gamma}_{\tau,\tau}$*

$$\sigma_n(f)_{\dot{F}^{\beta}_{p,t}} \leqslant C n^{-(\gamma-\beta)/d} \, \|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}. \tag{3.2}$$

Proof. Let $f \in \dot{B}^{\gamma}_{\tau,\tau}$ from (1.4) and (1.5) we have that

$$f = \sum_{I \in \mathscr{D}} A_I(f) \qquad \text{and} \qquad (a_I(f)\, |I|^{-\frac{\gamma}{d} + \frac{1}{\tau} - \frac{1}{2}})_I \in \ell_\tau.$$

Let now $\tilde{a}_I(f) := a_I(f)\, |I|^{-\frac{\gamma}{d} + \frac{1}{\tau} - \frac{1}{2}}$ and $M := \|\tilde{a}_I(f)\|_{\ell_\tau} = \|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}$. For every $j \in \mathbb{Z}$ we set

$$\Lambda_j := \{I : 2^{-j} < \tilde{a}_I(f) \leqslant 2^{-j+1}\}$$

and we define $S_j := \sum_{I \in \Lambda_j} A_I(f)$. We are going to approximate f by $T_k := \sum_{j \leqslant k} S_j$. Since $(\tilde{a}_I(f)) \in \ell_\tau$, it follows immediately that for every $\varepsilon > 0$

$$\#\{I : \tilde{a}_I(f) \geqslant \varepsilon\} \leqslant M^\tau \varepsilon^{-\tau},$$

and therefore for each $k \in \mathbb{Z}$ we have

$$\sum_{j \leqslant k} \#\Lambda_j \leqslant C M^\tau 2^{k\tau},$$

which shows that $T_k \in \Sigma_N$, with $N = [C M^\tau 2^{k\tau}]$. In order to prove (3.2) it suffices to establish that

$$\|f - T_k\|_{\dot{F}^{\beta}_{p,t}} \leqslant C (M^\tau 2^{k\tau})^{-(\gamma-\beta)/d} \, \|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}. \tag{3.3}$$

For general $n \in \mathbb{Z}_+$ the result will follow from the monotonicity of $\sigma_n(f)_{\dot{F}^{\beta}_{p,t}}$.

For (3.3) we shall consider two separate cases:

Case I: $t \geqslant p$. Taking into account that $1/\tau - \gamma/d = 1/p - \beta/d$ we get that

$$\begin{aligned}
\|f - T_k\|^{p}_{\dot{F}^{\beta}_{p,t}} &= \int \Big(\sum_{j \geqslant k+1} \sum_{I \in \Lambda_j} (a_I(f)\, |I|^{-\frac{\beta}{d} - \frac{1}{2}} \chi_I)^t \Big)^{\frac{p}{t}} dx \\
&\leqslant \int \sum_{j \geqslant k+1} \sum_{I \in \Lambda_j} (a_I(f)\, |I|^{-\frac{\beta}{d} - \frac{1}{2}} \chi_I)^p \, dx \\
&\leqslant C \sum_{j \geqslant k+1} 2^{-jp} \int \sum_{I \in \Lambda_j} (|I|^{-\frac{1}{p}} \chi_I)^p \, dx \\
&\leqslant C \sum_{j \geqslant k+1} 2^{-jp} \# \Lambda_j \leqslant C \sum_{j \geqslant k+1} M^{\tau} 2^{-j(p-\tau)} \\
&\leqslant C M^{\tau} 2^{-k(p-\tau)}.
\end{aligned}$$

Case II: $t < p$. It follows easily that

$$\begin{aligned}
\text{(3.4)} \qquad \|f - T_k\|^{p}_{\dot{F}^{\beta}_{p,t}} &= \int \Big(\sum_{j \geqslant k+1} \sum_{I \in \Lambda_j} (a_I(f)\, |I|^{-\frac{\beta}{d} - \frac{1}{2}} \chi_I)^t \Big)^{\frac{p}{t}} dx \\
&\leqslant \int \Big(\sum_{j \geqslant k+1} \sum_{I \in \Lambda_j} (2^{-j}\, |I|^{-\frac{1}{p}} \chi_I)^t \Big)^{\frac{p}{t}} dx.
\end{aligned}$$

Since $p > \tau$ we can find $\delta > 0$ sufficiently small such that $p(t-\delta)/t > \tau$. Using Minkowski's inequality we get

$$\begin{aligned}
\text{(3.5)} \quad & \Big(\sum_{j \geqslant k+1} \sum_{I \in \Lambda_j} (2^{-j}\, |I|^{-\frac{1}{p}} \chi_I)^t \Big)^{\frac{p}{t}} \\
&= \Big(\sum_{j \geqslant k+1} 2^{-j\delta} \sum_{I \in \Lambda_j} 2^{-j(t-\delta)}\, |I|^{-\frac{t}{p}} \chi_I \Big)^{\frac{p}{t}} \\
&\leqslant \Big(\sum_{j \geqslant k+1} 2^{-j\delta\left(\frac{p}{p-t}\right)} \Big)^{\frac{p}{t}\left(\frac{p-t}{p}\right)} \Big(\sum_{j \geqslant k+1} \Big(\sum_{I \in \Lambda_j} 2^{-j(t-\delta)}\, |I|^{-\frac{t}{p}} \chi_I \Big)^{\frac{p}{t}} \Big) \\
&\leqslant 2^{-k\delta p/t} \sum_{j \geqslant k+1} \Big(\sum_{I \in \Lambda_j} 2^{-j(t-\delta)}\, |I|^{-\frac{t}{p}} \chi_I \Big)^{\frac{p}{t}}.
\end{aligned}$$

If for each finite set of dyadic cubes Λ we let $I_\Lambda(x)$ denote the smallest cube in Λ containing x then from (3.4) and (3.5) we get

$$\begin{aligned}
\|f-T_k\|_{\dot F^\beta_{p,t}}^p &\leqslant C2^{-k\delta p/t}\int \sum_{j\geqslant k+1}\left(\sum_{I\in\Lambda_j} 2^{-j(t-\delta)}\,|I|^{-\frac{t}{p}}\chi_I\right)^{\frac{p}{t}}dx\\
&= C2^{-k\delta p/t}\sum_{j\geqslant k+1}2^{-jp(t-\delta)/t}\int\left(\sum_{I\in\Lambda_j}|I|^{-\frac{t}{p}}\chi_I\right)^{\frac{p}{t}}dx\\
&\leqslant C2^{-k\delta p/t}\sum_{j\geqslant k+1}2^{-jp(t-\delta)/t}\int |I_{\Lambda_j}(x)|^{-1}\,dx\\
&\leqslant C2^{-k\delta p/t}\sum_{j\geqslant k+1}2^{-jp(t-\delta)/t}\#\Lambda_j\\
&\leqslant C2^{-k\delta p/t}M^\tau\sum_{j\geqslant k+1}2^{-j(p(t-\delta)/t-\tau)}\leqslant CM^\tau 2^{-k(\delta p/t+p(t-\delta)/t-\tau)}\\
&\leqslant CM^\tau 2^{-k(p-\tau)}.
\end{aligned}$$

In either case we proved that

$$\|f-T_k\|_{\dot F^\beta_{p,t}}\leqslant CM^{\tau/p}2^{-k\left(1-\frac{\tau}{p}\right)}=C(M^\tau 2^{k\tau})^{-(\gamma-\beta)/d}\,\|f\|_{\dot B^\gamma_{\tau,\tau}}. \qquad \blacksquare$$

In the next theorem we study n-term approximation when the error is measured in the space $\dot B^\beta_{p,t}$, where $\beta\in\mathbb{R}$, $0<p<\infty$ and $0<t\leqslant\infty$.

THEOREM 3.6. *Let $0<p<\infty$, $0<t\leqslant\infty$ and $\beta<\gamma$. If τ, r are defined by $1/\tau-1/p=1/r-1/t=(\gamma-\beta)/d$. Then, for every $f\in\dot B^\gamma_{\tau,r}$*

$$\sigma_n(f)_{\dot B^\beta_{p,t}}\leqslant Cn^{-(\gamma-\beta)/d}\,\|f\|_{\dot B^\gamma_{\tau,r}}.$$

Proof. For any $\varepsilon>0$ and $i\in\mathbb{Z}$, we define

$$K_i(\varepsilon):=\left\{m\in\mathbb{Z}\colon 2^i\varepsilon<\left(\sum_{I\in\mathcal{D}_m}\tilde a_I(f)^\tau\right)^{1/\tau}\leqslant 2^{i+1}\varepsilon\right\},$$

where as before, $\tilde a_I(f):=|I|^{-\frac{\gamma}{d}+\frac{1}{\tau}-\frac{1}{2}}\alpha_I(f)$.

Applying the previous theorem to $f_m:=\sum_{I\in\mathcal{D}_m}A_I(f)$, $m\in\mathbb{Z}$, for each $n\in\mathbb{Z}_+$ we can find a set $\Lambda^m_n\subset\mathcal{D}_m$ with cardinality $\#\Lambda^m_n\leqslant n$, and such that $S^m_n:=\sum_{I\in\Lambda^m_n}A_I(f)\in\Sigma_n$ satisfies

$$\|f_m-S^m_n\|_{\dot F^\beta_{p,p}}\leqslant Cn^{-(\gamma-\beta)/d}\left\|\sum_{I\in\mathcal{D}_m}A_I(f)\right\|_{\dot B^\gamma_{\tau,\tau}},$$

i.e.,

$$\left(\sum_{I\notin\Lambda_n^m}\tilde a_I(f)^p\right)^{1/p}\leqslant Cn^{-(\gamma-\beta)/d}\left(\sum_{I\in\mathscr{D}_m}\tilde a_I(f)^\tau\right)^{1/\tau}. \tag{3.7}$$

For each $m\in K_i(\varepsilon)$ we let $S^m_{[2^{ir}]}$ be a near best approximant to f_m from $\Sigma_{[2^{ir}]}$ satisfying (3.7) and we define $T_\varepsilon:=\sum_{i\geqslant 0}\sum_{m\in K_i(\varepsilon)}S^m_{[2^{ir}]}$. It is easily seen that $T_\varepsilon\in\Sigma_{N_\varepsilon}$ with

$$N_\varepsilon\leqslant\sum_{i\geqslant 0}\#K_i(\varepsilon)\,2^{ir}.$$

Using that

$$\|f\|^r_{\dot B^\gamma_{\tau,r}}=\sum_{i\in\mathbb{Z}}\sum_{m\in K_i(\varepsilon)}\left(\sum_{I\in\mathscr{D}_m}\tilde a_I(f)^\tau\right)^{r/\tau}\geqslant C\sum_{i\in\mathbb{Z}}\#K_i(\varepsilon)(2^i\varepsilon)^r,$$

we get an upper estimate for N_ε, namely,

$$N_\varepsilon\leqslant C\varepsilon^{-r}\,\|f\|^r_{\dot B^\gamma_{\tau,r}}.$$

It follows that

$$\begin{aligned}\sigma_{N_\varepsilon}(f)^t_{\dot B^\beta_{p,t}}&\leqslant C\left(\left\|\sum_{i<0}\sum_{m\in K_i(\varepsilon)}f_m\right\|^t_{\dot B^\beta_{p,t}}+\left\|\sum_{i\geqslant 0}\sum_{m\in K_i(\varepsilon)}(f_m-S^m_{[2^{ir}]})\right\|^t_{\dot B^\beta_{p,t}}\right)\\&=:I_1^t+I_2^t.\end{aligned}$$

For I_1, using that $p>\tau$ and $t>r$ we get

$$\begin{aligned}I_1^t&\leqslant C\sum_{i<0}\sum_{m\in K_i(\varepsilon)}\left(\sum_{I\in\mathscr{D}_m}\tilde a_I(f)^\tau\right)^{t/\tau}\leqslant C\sum_{i<0}\#K_i(\varepsilon)(2^i\varepsilon)^t\\&\leqslant C\varepsilon^{t-r}\sum_{i<0}\#K_i(\varepsilon)(2^i\varepsilon)^r\\&\leqslant C\varepsilon^{t-r}\,\|f\|^r_{\dot B^\gamma_{\tau,r}}.\end{aligned} \tag{3.8}$$

Similarly for I_2, using (3.7) and that $1/r-1/t=(\gamma-\beta)/d$ we have

$$\begin{aligned}I_2^t&\leqslant C\sum_{i\geqslant 0}\sum_{m\in K_i(\varepsilon)}\left(\sum_{I\notin\Lambda^m_{[2^{ir}]}}\tilde a_I(f)^p\right)^{t/p}\\&\leqslant C\sum_{i\geqslant 0}\sum_{m\in K_i(\varepsilon)}2^{-irt(\gamma-\beta)/d}\left(\sum_{I\in\mathscr{D}_m}\tilde a_I(f)^\tau\right)^{t/\tau}\end{aligned} \tag{3.9}$$

$$\leqslant C \sum_{i \geqslant 0} \sum_{m \in K_i(\varepsilon)} 2^{-i(t-r)} 2^{it} \varepsilon^t$$

$$\leqslant C \sum_{i \geqslant 0} \# K_i(\varepsilon)\, 2^{ir} \varepsilon^t$$

$$\leqslant C \varepsilon^{t-r} \, \|f\|^r_{\dot{B}^\gamma_{\tau, r}}.$$

From (3.8) and (3.9) we immediately get that for every $\varepsilon > 0$

$$\sigma_{N_\varepsilon}(f)_{\dot{B}^\beta_{p,t}} \leqslant C(\varepsilon^{-r} \, \|f\|^r_{\dot{B}^\gamma_{\tau, r}})^{-(\gamma-\beta)/d} \, \|f\|_{\dot{B}^\gamma_{\tau, r}}. \tag{3.10}$$

In order to extend (3.10) to general $n \in \mathbb{Z}_+$ one has to choose $\varepsilon := n^{-1/r} \, \|f\|_{\dot{B}^\gamma_{\tau, r}}$ and as in the previous theorem, to use the monotonicity of $\sigma_n(f)_{\dot{B}^\beta_{p,t}}$. ∎

4. BERNSTEIN-TYPE INEQUALITIES

In this section we are going to establish Bernstein-type inequalities regarding the Triebel–Lizorkin and Besov spaces.

THEOREM 4.1. *Let $0 < p < \infty$, $0 < t \leqslant \infty$ and $\beta < \gamma$. If τ is defined by $1/\tau = (\gamma - \beta)/d + 1/p$. Then, for every $S \in \Sigma_n$*

$$\|S\|_{\dot{B}^\gamma_{\tau, \tau}} \leqslant C n^{(\gamma-\beta)/d} \, \|S\|_{\dot{F}^\beta_{p,t}}.$$

Proof. We recall that for each finite set of dyadic cubes Λ, $I_\Lambda(x)$ is smallest cube in Λ containing x. Then, if $S = \sum_{I \in \Lambda} A_I(S)$

$$\|S\|^\tau_{\dot{B}^\gamma_{\tau, \tau}} = \sum_{I \in \Lambda} (|I|^{-\frac{\gamma}{d} + \frac{1}{\tau} - \frac{1}{2}} a_I(S))^\tau$$

$$= \int \sum_{I \in \Lambda} |I|^{\frac{\tau(\beta-\gamma)}{d}} (|I|^{-\frac{\beta}{d} - \frac{1}{2}} a_I(S))^\tau \chi_I(x) \, dx$$

$$\leqslant \int (S^\beta_t(S, x))^\tau \sum_{I \in \Lambda} |I|^{\frac{\tau(\beta-\gamma)}{d}} \chi_I(x) \, dx$$

$$\leqslant \|S_t^\beta(S,\cdot)\|_{L_p}^\tau \left(\int\left(\sum_{I\in\Lambda}|I|^{\frac{\tau(\beta-\gamma)}{d}}\chi_I(x)\right)^{\frac{p}{p-\tau}}dx\right)^{\frac{p-\tau}{p}}$$

$$\leqslant C\,\|S_t^\beta(S,\cdot)\|_{L_p}^\tau \left(\int |I_\Lambda(x)|^{\frac{\tau p(\beta-\gamma)}{d(p-\tau)}}\,dx\right)^{\frac{p-\tau}{p}}$$

$$= C\,\|S_t^\beta(S,\cdot)\|_{L_p}^\tau \left(\int |I_\Lambda(x)|^{-1}\,dx\right)^{\frac{p-\tau}{p}}$$

$$\leqslant C(\#\Lambda)^{\frac{p-\tau}{p}}\,\|S\|_{\dot F_{p,t}^\beta}^\tau = Cn^{\tau(\gamma-\beta)/d}\,\|S\|_{\dot F_{p,t}^\beta}^\tau,$$

where in the second inequality we used Hölder's inequality. ∎

A similar theorem holds for Besov spaces as well:

THEOREM 4.2. *Let $0<p<\infty$, $0<t\leqslant\infty$ and $\beta<\gamma$. If τ, r are defined by $1/\tau-1/p=1/r-1/t=(\gamma-\beta)/d$. Then, for every $S\in\Sigma_n$*

$$\|S\|_{\dot B_{\tau,r}^\gamma}\leqslant Cn^{(\gamma-\beta)/d}\,\|S\|_{\dot B_{p,t}^\beta}.$$

Proof. Let $S=\sum_{I\in\Lambda}A_I(S)\in\Sigma_n$. For every $m\in\mathbb{Z}$, we define $\Lambda_m:=\Lambda\cap\mathscr{D}_m$, then using Hölder's inequality we get

$$\|S\|_{\dot B_{\tau,r}^\gamma}^r = \sum_{m\in\mathbb{Z}}\left(\sum_{\Lambda_m}\left(|I|^{-\frac{\gamma}{d}-\frac{1}{2}+\frac{1}{\tau}}a_I(S)\right)^\tau\right)^{\frac{r}{\tau}}$$

$$= \sum_{m\in\mathbb{Z}}\left(\sum_{\Lambda_m}\left(|I|^{-\frac{\beta}{d}+\frac{1}{2}+\frac{1}{p}}a_I(S)\right)^\tau\right)^{\frac{r}{\tau}}$$

$$\leqslant \sum_{m\in\mathbb{Z}}(\#\Lambda_m)^{r\left(\frac{1}{\tau}-\frac{1}{p}\right)}\left(\sum_{\Lambda_m}\left(|I|^{-\frac{\beta}{d}-\frac{1}{2}+\frac{1}{p}}a_I(S)\right)^p\right)^{r/p}$$

$$\leqslant\left(\sum_{m\in\mathbb{Z}}(\#\Lambda_m)^{r\left(\frac{1}{\tau}-\frac{1}{p}\right)\left(r\left(\frac{1}{r}-\frac{1}{t}\right)\right)^{-1}}\right)^{1-\frac{r}{t}}\left(\sum_{m\in\mathbb{Z}}\left(\sum_{\Lambda_m}\left(|I|^{-\frac{\beta}{d}-\frac{1}{2}+\frac{1}{p}}a_I(S)\right)^p\right)^{t/p}\right)^{r/t}$$

$$=\left(\sum_{m\in\mathbb{Z}}\#\Lambda_m\right)^{r\left(\frac{1}{r}-\frac{1}{t}\right)}\|S\|_{\dot B_{p,t}^\beta}^r = n^{r(\gamma-\beta)/d}\,\|S\|_{\dot B_{p,t}^\beta}^r.\quad ∎$$

5. PROOFS OF THEOREMS 1.9 AND 1.16

Having established the Bernstein and Jackson-type inequalities in Sections 3 and 4 respectively the proofs of both theorems follow immediately as a direct application of Theorem 2.1.

6. PROOFS OF THEOREM 1.11 AND COROLLARY 1.14

The proof of Theorem 1.11 will be primarily based on the following embeddings: Let $p,\ \alpha > 0,\ \beta \in \mathbb{R}$ and τ be such that $1/\tau = 1/p + a/d$. If $\tilde{\tau} := \min\{1, \tau\}$ then

$$A_{\tilde{\tau}}^{\alpha/d}(\dot{F}_{p,t}^{\beta}) \subset \dot{B}_{\tau,\tau}^{\beta+\alpha} \subset A_{\infty}^{\alpha/d}(\dot{F}_{p,t}^{\beta}). \tag{6.1}$$

To prove the right side of (6.1) we note from Jackson's inequality (Theorem 3.1) that $|f|_{A_{\infty}^{\alpha/d}(\dot{F}_{p,t}^{\beta})} \leqslant C\, \|f\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}}$. In addition if $S_0 \in \Sigma_1$, is a near best approximant to f form Σ_1, then

$$\|S_0\|_{\dot{F}_{p,t}^{\beta}} = \|S_0\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}} \leqslant C\, \|f\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}}$$

which implies that,

$$\|f\|_{A_{\infty}^{\alpha/d}(\dot{F}_{p,t}^{\beta})} \leqslant C\, \|f\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}}.$$

As far as the left embedding is concerned for every $k \in \mathbb{Z}_+$, we let $S_k \in \Sigma_{2^k}$ be such that

$$\|f - S_k\|_{\dot{F}_{p,t}^{\beta}} \leqslant 2\sigma_{2^k}(f)_{\dot{F}_{p,t}^{\beta}}$$

and $S_{-1} := 0$, then using the Bernstein-type inequality (Theorem 4.1) we have

$$\begin{aligned}
\|f\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}}^{\tilde{\tau}} &\leqslant \sum_{k=0}^{\infty} \|S_k - S_{k-1}\|_{\dot{B}_{\tau,\tau}^{\beta+\alpha}}^{\tilde{\tau}} \\
&\leqslant C \sum_{k=0}^{\infty} 2^{k\tilde{\tau}\alpha/d}\, \|S_k - S_{k-1}\|_{\dot{F}_{p,t}^{\beta}}^{\tilde{\tau}} \\
&\leqslant C \left(\sum_{k=0}^{\infty} 2^{k\tilde{\tau}\alpha/d} \sigma_{2^k}(f)_{\dot{F}_{p,t}^{\beta}}^{\tilde{\tau}} + \|S_0\|_{\dot{F}_{p,t}^{\beta}}^{\tilde{\tau}} \right) \\
&= C\, \|f\|_{A_{\tilde{\tau}}^{\alpha/d}(\dot{F}_{p,t}^{\beta})}^{\tilde{\tau}}.
\end{aligned}$$

From (6.1) it follows that for every $0 < \theta < 1$ and $q > 0$, if $1/\tau_1 = \alpha_1/d + 1/p$ and $1/\tau_2 = \alpha_2/d + 1/p$, then

$$\begin{aligned}
(\mathscr{A}_{\tilde{\tau}_1}^{\alpha_1/d}(\dot{F}_{p,t}^{\beta}), \mathscr{A}_{\tilde{\tau}_2}^{\alpha_2/d}(\dot{F}_{p,t}^{\beta}))_{\theta,q} &\subset (\dot{B}_{\tau_1,\tau_1}^{\beta+\alpha_1}, \dot{B}_{\tau_2,\tau_2}^{\beta+\alpha_2})_{\theta,q} \\
&\subset (\mathscr{A}_{\infty}^{\alpha_1/d}(\dot{F}_{p,t}^{\beta}), \mathscr{A}_{\infty}^{\alpha_2/d}(\dot{F}_{p,t}^{\beta}))_{\theta,q}.
\end{aligned}$$

However, since the approximation spaces are interpolation spaces as well (see [DP1]) we have from the reiteration theorem (see [DL]) that

$$(\mathscr{A}_{\tilde{\tau}_1}^{\alpha_1/d}(\dot{F}_{p,t}^{\beta}), \mathscr{A}_{\tilde{\tau}_2}^{\alpha_2/d}(\dot{F}_{p,t}^{\beta}))_{\theta,q} = \mathscr{A}_{q}^{\alpha/d}(\dot{F}_{p,t}^{\beta}) = (\mathscr{A}_{\infty}^{\alpha_1/d}(\dot{F}_{p,t}^{\beta}), \mathscr{A}_{\infty}^{\alpha_2/d}(\dot{F}_{p,t}^{\beta}))_{\theta,q},$$

where $\alpha = (1-\theta)\,\alpha_1 + \theta\alpha_2$. In other words for every $t > 0$

$$\mathscr{A}_{q}^{\alpha/d}(\dot{F}_{p,t}^{\beta}) = (\dot{B}_{\tau_1,\tau_1}^{\beta+\alpha_1}, \dot{B}_{\tau_2,\tau_2}^{\beta+\alpha_2})_{\theta,q},$$

which establishes (1.12).

On the other hand in order to prove (1.13), if $1/\tau = \gamma/d + 1/p$ and $0 < \alpha < \gamma$, from Lemma 6.2 below, we have

$$(\dot{B}_{p,p}^{\beta}, \dot{B}_{\tau,\tau}^{\beta+\gamma})_{\frac{\alpha}{\gamma},\mu} = \dot{B}_{\mu,\mu}^{\beta+\alpha}, \qquad \frac{1}{\mu} = \frac{\alpha}{d} + \frac{1}{p}.$$

From this fact and the reiteration theorem, if $1/\tau_1 = \alpha_1/d + 1/p$ and $1/\tau_2 = \alpha_2/d + 1/p$, $0 < \alpha_1, \alpha_2 < \gamma$ we get that for $0 < \theta < 1$ and $\alpha = (1-\theta)\,\alpha_1 + \theta\alpha_2$

$$(\dot{B}_{\tau_1,\tau_1}^{\beta+\alpha_1}, \dot{B}_{\tau_2,\tau_2}^{\beta+\alpha_2})_{\theta,q} = \dot{B}_{q,q}^{\beta+\alpha}, \qquad \frac{1}{q} = \frac{\alpha}{d} + \frac{1}{p}.$$

This concludes the proof of Theorem 1.11.

Finally, we note that Corollary 1.14 is a direct application of (1.10) and (1.13).

LEMMA 6.2. *Let $\beta \in \mathbb{R}$, $0 < p, \gamma < \infty$ and $1/\tau = \gamma/d + 1/p$. If $0 < \alpha < \gamma$, and $1/\mu = \alpha/d + 1/p$ then*

$$(\dot{B}_{p,p}^{\beta}, \dot{B}_{\tau,\tau}^{\beta+\gamma})_{\frac{\alpha}{\gamma},\mu} = \dot{B}_{\mu,\mu}^{\beta+\alpha}.$$

Proof. Let T be the linear mapping that maps a function f to its wavelet coefficients according to

$$T: f \to (|I|^{-\beta/d+1/p-1/2}\,(a_I^v(f))_{v\in V})_I.$$

We note that for every $f \in \dot{B}_{p,p}^{\beta}$,

$$\|Tf\|^p_{\ell_p(\mathcal{D}\times V)} = \sum_{I\in\mathcal{D}} \sum_{v\in V} (|I|^{-\beta/d+1/p-1/2}\,|a^v_I(f)|)^p$$

$$\approx \sum_{I\in D} (|I|^{-\beta/d+1/p-1/2}\,a_I(f))^p = \|f\|^p_{\dot{B}^\beta_{p,p}}$$

and similarly, since $1/\tau=\gamma/d+1/p$,

$$\|Tf\|^\tau_{\ell_\tau(\mathcal{D}\times V)} = \sum_{I\in\mathcal{D}} \sum_{v\in V} (|I|^{-\beta/d+1/p-1/2}\,|a^v_I(f)|)^\tau$$

$$\approx \sum_{I\in\mathcal{D}} (|I|^{-(\beta+\gamma)/d+1/\tau-1/2}\,a_I(f))^\tau = \|f\|^\tau_{\dot{B}^{\beta+\gamma}_{\tau,\tau}}.$$

It follows, that $f\in(\dot{B}^\beta_{p,p}, \dot{B}^{\beta+\gamma}_{\tau,\tau})_{\theta,\mu}$ if and only if $Tf\in(\ell_p,\ell_\tau)_{\theta,\mu}$. Taking into account that for $1/\mu=(1-\theta)/p+\theta/\tau$, $(\ell_p,\ell_\tau)_{\theta,\mu}=\ell_\mu$ (see [BL, p. 109]) we obtain that for this particular case $f\in(\dot{B}^\beta_{p,p}, \dot{B}^{\beta+\gamma}_{\tau,\tau})_{\theta,\mu}$, if and only if $Tf\in\ell_\mu$. This concludes the proof of the lemma in view of the fact that for $\theta=\alpha/\gamma$ we have $1/\mu=\alpha/d+1/p$ and

$$\|Tf\|_{\ell_\mu} = \left(\sum_{I\in D} (|I|^{-\beta/d+1/p-1/2}\,a_I(f))^\mu\right)^{1/\mu} = \|f\|_{\dot{B}^{\beta+\alpha}_{\mu,\mu}}. \qquad \blacksquare$$

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